stichting
mathematisch
centrum

MC

AFDELING MATHEMATISCHE BESLISKUNDE BW 157/82 MAART
(DEPARTMENT OF OPERATIONS RESEARCH)

H. NIJMEIJER & A. VAN DER SCHAFT

THE DISTURBANCE DECOUPLING PROBLEM FOR NONLINEAR CONTROL SYSTEMS

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 93C10, 58F35, 49E05

The Disturbance Decoupling Problem for nonlinear control systems*)

by

Henk Nijmeijer & Arjan van der Schaft**)

ABSTRACT

Necessary and sufficient conditions are derived for the solution of the Disturbance Decoupling Problem for general nonlinear control systems. Some conceptual algorithms needed are discussed.

KEY WORDS & PHRASES: *nonlinear control systems, invariant distributions, Disturbance Decoupling Problem, connections*

---

*) This paper will be submitted for publication elsewhere.

**) Mathematical Institute, University of Groningen, Groningen, the Netherlands

1. INTRODUCTION

Consider the linear system

$$(1.1)\qquad \begin{cases} \dot{x} = Ax + Bu + Eq \\ z = Hx \end{cases}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, disturbance $q \in \mathbb{R}^r$ and the to-be-controlled variable $z \in \mathbb{R}^p$. A, B, E and H are matrices of appropriate dimensions. The Disturbance Decoupling Problem (D.D.P.) consists of finding a state feedback $u = Fx + v$ which decouples the disturbance from the to-be-controlled variable z. Equivalently, after feedback the transfer function from q to z has to be zero. The solvability of D.D.P. can be constructively checked in the following way (cf. [8]).

(i) Construct the maximal controlled invariant subspace in the kernel of H: $V^*_{\ker H}$.

(ii) Check if $\operatorname{Im} E \subset V^*_{\ker H}$.

Recently a similar theory has been developed for nonlinear systems where the inputs and the disturbances enter linearly in the equations (cf. [2,3]).

$$(1.2)\qquad \begin{cases} \dot{x} = A(x) + \sum_{i=1}^{m} B_i(x)u_i + \sum_{j=1}^{r} E_j(x)q_j \\ z = H(x). \end{cases}$$

The procedure is the same: construct the maximal controlled invariant distribution contained in $\ker dH$, call this $D^*_{\ker dH}$. Then D.D.P. is locally solvable if and only if $\operatorname{span}\{E_1,\dots,E_r\} \subset D^*_{\ker dH}$. Applications of these results can be found in [1,7].

In our previous paper [5] we treated controlled invariance for a general nonlinear system $\dot{x} = f(x,u)$. With the aid of this we can treat the D.D.P. for the system

$$(1.3)\qquad \begin{cases} \dot{x} = f(x,u) + \sum_{j=1}^{r} E_j(x)q_j \\ z = H(x). \end{cases}$$

In fact, D.D.P. is locally solvable if and only if there exists a controlled invariant distribution D (w.r.t. $\dot{x} = f(x,u)$) such that $\text{span}\{E_1,\ldots,E_r\} \subset D$.

In this paper we will treat the most general case where also the disturbances enter in a nonlinear way:

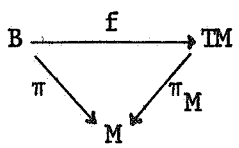
$$(1.4)\qquad \begin{cases} \dot{x} = f(x,u,q) \\ z = H(x). \end{cases}$$

To give a coordinate free description of the Disturbance Decoupling Problem in this case we first have to generalize the definition of a control system as in [5], to the definition of a control system with disturbances. Then the local solution will readily follow.

Furthermore, just as in the linear case, we will give some algorithms for checking solvability of D.D.P. (section 3).

2. CONTROLLED INVARIANCE FOR NONLINEAR CONTROL SYSTEMS WITH DISTURBANCES

As in our previous paper [5] we use the following setting for a nonlinear control system. Let M be a smooth n-dimensional manifold, denoting the state space. Let $\pi: B \to M$ be a smooth fiber bundle, whose fibers represent the state-dependent input spaces. Then a *control system* $\Sigma(M,B,f)$ is defined by the commutative diagram



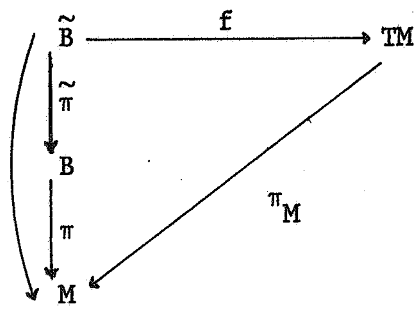
where TM denotes the tangent bundle of M, with natural projection π_M and f is a smooth map.

In local coordinates x for M, (x,u) for B, this coordinate free definition comes down to $\dot{x} = f(x,u)$.

We now want to formalize the situation that our control system also contains disturbances (which also can be interpreted as unknown inputs). This leads to the following definition:

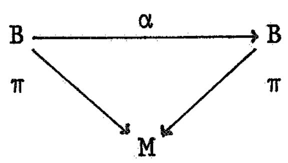
<u>DEFINITION 2.1</u>. *A control system with disturbances* $\Sigma = \Sigma(M,B,\tilde{B},f)$ is given

by the following. Let $\Sigma(M,\tilde{B},f)$ be a control system. Let $\tilde{\pi}\colon \tilde{B} \to B$ and $\pi\colon B \to M$ be fiber bundles, where the fibers of $\pi\colon B \to M$ represent the state-dependent input spaces and the fibres of $\tilde{\pi}\colon \tilde{B} \to B$ represent the state- and input-dependent disturbance spaces. If we let $\pi' := \pi \circ \tilde{\pi}$ then the fibers of the bundle $\pi'\colon \tilde{B} \to M$ represent the state-dependent input *and* disturbance spaces. So a control system with disturbances is given by the following commutative diagram

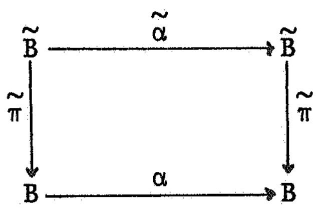


In local coordinates x for M, (x,u) for B (u for the inputs) and (x,u,q) for $\tilde{B}$ (q for the disturbances) this definition simple comes down to $\dot{x} = f(x,u,q)$.

In this framework *state-feedback* is given by the following procedure: Let α be a fiber-preserving diffeomorphism on B such that the diagram



commutes. Consider an *arbitrary* fiber preserving diffeomorphism $\tilde{\alpha}$ on $\tilde{B}$ such that we also have that the diagram



commutes. Then the system $\Sigma(M,B,\tilde{B},f)$ after *state feedback* $\tilde{\alpha}$ is given by $\Sigma(M,B,\tilde{B},\hat{f})$ with $\hat{f} = f\circ\alpha$ (compare with [6]).

REMARK. In local coordinates this means that the system $\dot{x} = f(x,u,q)$ is modified by the state feedback $(u,q) = \tilde{\alpha}(x,v,q') = (\alpha(x,v),\tilde{q}(x,v,q'))$ to the system $\dot{x} = f(x,\alpha(x,v),\tilde{q}(x,v,q'))$, where $\tilde{q}(x,v,\cdot)\colon \tilde{\pi}^{-1}(x,v) \to \tilde{\pi}^{-1}(x,v)$ is an arbitrary diffeomorphism (induced by $\tilde{\alpha}$).

From the above discussion of this notion of state feedback and from [5], the next definition should be clear:

DEFINITION 2.2. An involutive distribution D, of fixed dimension, on M, is *locally controlled invariant for the control system with disturbances* $\Sigma(M,B,\tilde{B},f)$, if locally around each point $x_0 \in M$ there exist fiber respecting coordinates (x,u) for B such that for all fiber respecting coordinates (x,u,q) for $\tilde{B}$ we have that for every fixed u and q $[f(\cdot,u,q),D] \subset D$.

REMARK. This implies that for every time function $\bar{u}(\cdot)$ and $\bar{q}(\cdot)$ also $[f(\cdot,\bar{u},\bar{q}),D] \subset D$.

What are the conditions that a distribution D is locally controlled invariant for the control system with disturbances? The next theorem, which is a combination of the results of [5] and [6] exactly yields the solution.

THEOREM 2.3. *Let* $\Sigma = \Sigma(M,B,\tilde{B},f)$ *be a control system with disturbances. Let* $Q := \operatorname{Ker} \tilde{\pi}_*$ *and* $R := \operatorname{Ker} \pi'_*$. *Then an involutive distribution D of fixed dimension, is locally controlled invariant for the control system with disturbances if and only if the following three conditions hold.*

(i) $f_*(\pi'^{-1}_*(D)) \subset \mathring{D} + f_*(R)$

(ii) $f_*(Q) \subset \mathring{D}$

(iii) $\mathring{D} + f_*(R)$ *and* $f_*(Q)$ *have fixed dimension.*

REMARK. For the definition of $\mathring{D}$ we refer to [5] or [6].

PROOF OF THEOREM 2.3. The proof resembles that of theorem 3.1 of [6]. We note that from (i) it follows that we can locally construct a state feedback for the system $\Sigma(M,\tilde{B},f)$ (cf. [5]). But in principle this feedback depends on the input space of the bundle $\pi': \tilde{B} \to M$; i.e. the state feedback can also depend on the disturbances. Following [5] we know that the condition (i) is also equivalent to the existence of a distribution $\tilde{D}_{lift}$ on $\tilde{B}$ generated by an integrable connection on the bundle $\pi': \tilde{B} \to M$. In local coordinates this distribution $\tilde{D}_{lift}$ is generated by the vector fields:

$$(2.1) \qquad \frac{\partial}{\partial x_i} + h_i(x,u,q)\frac{\partial}{\partial u} + g_i(x,u,q)\frac{\partial}{\partial q}, \qquad i = 1,\dots,k$$

whereas D is generated by the vector fields (Frobenius)

$$(2.2) \qquad \frac{\partial}{\partial x_i}, \qquad i = 1,\dots,k$$

and the coefficients h_i and g_i in (2.1) satisfy certain integrability conditions (equation (4.30) of [5]). Now the second condition, (ii), in fact implies that we are able to choose the coefficients h_i in (2.1) such that h_i does not depend on q. Namely as in [6] we have that

$$(2.3) \qquad \tilde{D}_{lift} + Q = \mathrm{Span}\{\frac{\partial}{\partial x_i} + h_i(x,u,q)\frac{\partial}{\partial u}, \frac{\partial}{\partial q};\ i = 1,\dots,k\}$$

and then from (ii) it follows that

$$(2.4) \qquad [\tilde{D}_{lift}, Q] \subset \tilde{D}_{lift} + Q.$$

Now (2.4) - which is equivalent to the fact that $\tilde{\pi}_*(\tilde{D}_{lift})$ is a well-defined distribution on B - implies after an easy computation that $h_i(x,u,q)$ is independent from q! Knowing this we locally can construct a state feedback independent of q for $\Sigma(M,B,\tilde{B},f)$ similarly as in [6]. □

3. ALGORITHMS

In this section we will prove that every involutive distribution on the state space contains a maximal locally controlled invariant distribution. Furthermore we will give (conceptual) algorithms to compute this maximal locally controlled invariant distribution, and apply these to the general Disturbance Decoupling Problem. First we will start with the affine system locally given by

$$(3.1) \qquad \dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x), \qquad x \in M \text{ (a manifold)}$$

Following [4] we define

$$\Delta(x) := A(x) + \operatorname{span}\{B_1(x),\dots,B_m(x)\}$$

$$\Delta_0(x) := \operatorname{span}\{B_1(x),\dots,B_m(x)\}$$

and $\Delta^{-1}(\Delta_0 + D) :=$ {X a smooth vector field on M such that $[\Delta,X] \subset \Delta_0 + D$}. Then we can state

<u>THEOREM 3.1.</u>

a. *Let* D_1 *and* D_2 *be controlled invariant distributions on* M *for the affine system* (3.1). *Then* $\overline{D_1+D_2}$ *(the involutive closure of* $D_1 + D_2$*) is again controlled invariant.*

b. *Let* K *be an involutive distribution on* M *of dimension* k. *Then* K *contains a maximal controlled invariant distribution. Moreover, define*

$$D^0 = K$$

$$D^{m+1} = D^m \cap \Delta^{-1}(D^m + \Delta_0), \qquad m = 0,1,\dots .$$

Then $\lim_{m\to\infty} D^m = D^k$, *and when we assume that* $\bar{D}^k$ *has fixed dimension* $\bar{D}^k$ *is the maximal controlled invariant distribution in* K.

<u>PROOF.</u>

a) The essential part in the proof of a) is the Jacobi identity $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (see [2,3]).

b) From a) it follows that K contains a maximal locally controlled invariant distribution (see [2,3]). The algorithm above is given in [4] (see for a related algorithm [3]).

Next we will consider the situation for a general nonlinear system $\Sigma(M,B,f)$.

We define the extended system (see [5] for references) of $\Sigma(M,B,f)$ as the affine system on B given by:

$$\Delta^e(x,u) := \{X \in T_{(x,u)}B \mid \pi_* X = f(x,u)\}$$

$$\Delta_0^e(x,u) := \{X \in T_{(x,u)}B \mid \pi_* X = 0\}$$

i.e. in local coordinates simple

$$\begin{cases} \dot{x} = f(x,u) \\ \dot{u} = v \end{cases}$$

(v is the new input).

THEOREM 3.2. *Let D^1 and D^2 be locally controlled invariant distribution on M for the system $\Sigma(M,B,f)$ (see section 2). Then $\overline{D^1 + D^2}$ is also locally controlled invariant. Therefore, given an involutive distribution K on M there exists a maximal locally controlled invariant distribution contained in K.*

PROOF. From [5] we know that there exist involutive distributions D^1_{lift} and D^2_{lift} on B such that

$$\pi_* D^i_{lift} = D^i \qquad i = 1,2. \tag{3.2}$$

$$[\Delta^e, D^i_{lift}] \subset D^i_{lift} + \Delta_0^e \qquad i = 1,2. \tag{3.3}$$

When we define $D := \overline{D^1_{lift} + D^2_{lift}}$ it is clear from (3.2) that $\pi_* D = \overline{D^1 + D^2}$. From (3.3) it follows, by using the Jacobi-identity that $[\Delta^e, D] \subset D + \Delta_0^e$. Therefore $\overline{D_1 + D_2}$ is locally controlled invariant (the connection above $\overline{D_1 + D_2}$ is determined by D).

The algorithmic side becomes very simple by reducing it to the extended system:

ALGORITHM 3.3. Let K be an involutive distribution on M. Consider the extended system (Δ^e, Δ_0^e) of $\Sigma(M,B,f)$ and define the following distributions on B:

$$D^0 = K + \Delta_0^e$$

$$D^{m+1} = D^m \cap \Delta^{e^{-1}}(D^m + \Delta_0^e), \qquad m = 0,1,\dots .$$

Then $\lim_{m\to\infty} D^m = D^k$ (k is the dimension of D^0), and when we assume that $\bar{D}^k$ has fixed dimension, $\bar{D}^k$ is the maximal locally controlled invariant distribution in D^0 for the extended system. Furthermore because $[\Delta_0^e, \bar{D}^k] \subset \Delta_0^e + \bar{D}^k$, $\pi_*\bar{D}^k$ is a well defined distribution on M. In fact $\pi_*\bar{D}^k$ is the maximal locally controlled invariant distribution for $\Sigma(M,B,f)$ contained in K.

PROOF. The algorithm is just the algorithm of Theorem 3.1 for the (affine) extended system. That $\pi_*\bar{D}^k$ is the maximal controlled invariant distribution contained in K follows from the one-to-one correspondence between locally controlled invariant distributions of $\Sigma(M,B,f)$ and its extended system (see [5]).

REMARKS.

(i) (Compare [8, exercise 4.6].) Notice that while the algorithm 3.3 applies at first instance to the case that we have an output function $H: M \to Z$ and $K = \ker dH$, it can also be applied to the case that $H: B \to Z$. For instance, we can consider the disturbance decoupling problem where the to-be-controlled variable z equals $H(x,u)$. In this case we only have to change in the algorithm $D^0 = K + \Delta_0^e$ to $D^0 = \ker dH$, with $H: B \to Z$.

(ii) Note that theorem 3.2 is not valid for measured controlled invariance (i.e. controlled invariance by static output feedback, see [6]). In fact in general a maximal measured controlled invariant distribution does not exist.

COROLLARY 3.4. *Consider a control system with disturbances* $\Sigma(M,B,\tilde{B},f)$ *(see section 2). Let* $H: M \to Z$ *be a smooth function, with* $z = H(x)$ *the to-be-controlled variable. Then apply algorithm* 3.3 *to construct the maximal locally controlled invariant distribution contained in* $K := \ker dH$ *for the control system* $\Sigma(M,\tilde{B},f)$ *(i.e. we compute controlled invariance with respect to the whole input space R). Call this distribution* D. *Then the Disturbance Decoupling Problem is solvable if and only if* $f_*Q \subset \mathring{D}$, *or equivalently, if and only if* D *is locally controlled invariant for the system with disturbances* $\Sigma(M,B,\tilde{B},f)$.

REFERENCES

[1] CROUCH, P.E., N. CARMICHAEL, A. ISIDORI & S. MONACO, *Application of linear analytic systems theory to attitude control*, Report to E.S.T.E.C., by Applied Systems Studies, 1981.

[2] HIRSCHORN, R.M., (A,B)-*invariant distributions and disturbance decoupling of nonlinear systems*, SIAM J. Contr./Opt., 19, pp. 1-19, 1981.

[3] ISIDORI, A., A.J. KRENER, C. GORI-GIORGI & S. MONACO, *Nonlinear decoupling via feedback, a differential geometric approach*, IEEE Trans. Aut. Contr. 26, pp. 331-345, 1981.

[4] NIJMEIJER, H., *Controlled invariance for affine control systems*, Int. J. Contr. 34, pp. 825-833, 1981.

[5] NIJMEIJER, H. & A.J. VAN DER SCHAFT, *Controlled invariance for nonlinear systems*, to appear in IEEE Trans. Aut. Contr.

[6] NIJMEIJER, H. & A.J. VAN DER SCHAFT, *Controlled invariance by static output feedback for nonlinear systems*, to appear in Systems & Control Letters.

[7] NIJMEIJER, H. & A.J. VAN DER SCHAFT, *Controlled invariance for nonlinear systems: two worked examples*, submitted to IEEE Trans. Aut. Contr.

[8] WONHAM, W.M., *Linear multivariable control, a geometric approach*, second edition, Springer, 1979.